Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(02)00141-7
J. Appl. Maths Mechs, Vol. 66, No. 6, pp. 1001–1013, 2002

0021-8928/02/\$—see front matter

MOTION WITH A CONSTANT VELOCITY MODULUS IN A CENTRAL GRAVITATIONAL FIELD†

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(*Received* 10 *September* 2001)

The problem of the motion of a particle (point mass) with a constant velocity modulus in a Newtonian central gravitational field is investigated by two methods: using Lagrange's equations with a multiplier, and using the equations of dynamics proposed earlier [1] for systems with non-holonomic constraints that are non-linear with respect to velocities. A phase diagram of the motion is constructed. The structure of the trajectories as a function of the initial conditions is investigated. Formulae in the form of quadratures are obtained for calculating the time of motion along the trajectory and the angular distance of flight. A qualitative analysis of the properties of improper integrals expressing the angular distance is presented. These properties are illustrated by the results of a numerical investigation. The possibility of carrying out elementary manoeuvres in the vicinity of an attracting centre are analysed. © 2003 Elsevier Science Ltd. All rights reserved.

The problem of the motion of a particle (point mass) with a constant velocity modulus in a Newtonian central gravitational field is useful for studying the possibility of constructing different manoeuvres of spacecraft. Without dwelling on the methods for sustaining a constant absolute velocity of motion of a spacecraft, and assuming, for simplicity, that there is no change in its mass during a manoeuvre, the condition of constant velocity can be presented as a non-holonomic mechanical constraint on the motion of a particle that is non-linear with respect to the velocities. There was several versions of the procedure for obtaining the equations of dynamics for problems of this class. For example, on the assumption that the constraint for the problem in question is ideal, equations of motion with a Lagrange multiplier in spherical coordinates were obtained in [2]; however, their analytical solution was not constructed.

A version of the general equations of dynamics in Lagrangian coordinates has been proposed [1] for systems with ideal non-holonomic constraints that are non-linear with respect to the velocities. This version does not contain Lagrange multipliers, and the procedure used to set up the equations of dynamics generalizes the procedure for compiling Voronets equations. In a number of cases (for example, non-linear analogues of Chaplygin systems and, in particular, the problem under examination), this version of the general equations of dynamics may prove to be preferable, which is demonstrated below.

The problem of the motion of a particle with a constant velocity in a central field, unlike the standard problem of the passive motion of a particle in a central gravitational field, has no vector integral of the kinetic moment. Nor does it contain an integral of energy. In addition, as will be shown below, in the problem in question the direction of the kinetic moment vector is retained, and, by selecting a suitable system of coordinates, this problem can be reduced to an analogue of Chaplygin's system. As a result, it is possible to obtain an additional first integral and to carry out a fairly detailed qualitative analysis of the motion of a particle.

In the present paper we show that to each value of velocity there corresponds a circular orbit separating the set of trajectories having an infinitely remote point from the set of trajectories contained entirely within the given orbit. The family of all trajectories is invariant under group of rotations about the attracting centre. There are trajectories that *coil* with an infinite number of turns into a circular orbit both from inside and from outside the orbit with respect to the attracting centre. Furthermore, there are trajectories with any finite number of turns about the attracting centre. If such an orbit comes from infinity, then, after completing a given number of turns, it again departs to infinity without attaining a circular orbit. Similarly, trajectories coming from the side of the attracting centre return again to the attracting centre without attaining a circular orbit.

†*Prikl. Mat. Mekh.* Vol. 66, No. 6, pp. 1052–1065, 2002.

1. THE GENERAL EQUATIONS OF DYNAMICS

Suppose that, in space R^3, the configuration of a system of N particles taking into account all geometric constraints (mechanical and servo constraints), is uniquely defined by the coordinates $q_1, \dots, q_n$, $n \leq 3N$, such that the radius vectors of all particles of the system are expressed by the functions $\mathbf{r}_\gamma = \mathbf{r}_\gamma(q_1, \dots, q_n, t)$ $(\gamma = 1, \dots, N)$.

Let us assume that differential constraints

$$\phi_j(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = 0, \quad j = 1, \dots, m \tag{1.1}$$

are imposed on the system. Suppose that, as usual

$$T = \frac{1}{2}\sum_{\gamma=1}^{N} m_\gamma \mathbf{v}_\gamma^2, \quad Q_i = \sum_{\gamma=1}^{N} \mathbf{F}_\gamma \frac{\partial \mathbf{r}_\gamma}{\partial q_i}, \quad i = 1, \dots, n$$

are the kinetic energy and the generalized forces of the system, where m_γ is the mass, $\mathbf{v}_\gamma$ is the velocity of particles of the system and $\mathbf{F}_\gamma$ represents active forces. Assuming that the constraints are ideal, it is possible to use the D'Alembert–Lagrange principle

$$\sum_{i=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - Q_i\right]\delta q_i = 0$$

for any virtual displacements $\{\delta q_i, i = 1, \dots, n\}$ defined by the system of equations [3]

$$\sum_{i=1}^{n} \frac{\partial \phi_j}{\partial \dot{q}_i}\delta q_i = 0, \quad j = 1, \dots, m$$

The equations of dynamics with Lagrange multipliers λ_j will take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + \sum_{j=1}^{m} \lambda_j \frac{\partial \phi_j}{\partial \dot{q}_i}, \quad i = 1, \dots, n \tag{1.2}$$

The multipliers λ_j are found using constraint equations (1.1).

We will give the version of the equations of dynamics with excluded Lagrange multipliers [1, 4]. It will be assumed that the rank of the Jacobi matrix

$$\frac{\partial(\phi_1, \dots, \phi_m)}{\partial(\dot{q}_1, \dots, \dot{q}_n)}$$

is equal to m. This means that the system of constraints isolates in velocity space $\dot{q}_1, \dots, \dot{q}_n$ a surface of dimensionality $n - m$. We will represent it in the parametric form

$$\dot{q}_i = \dot{q}_i(q_1, \dots, q_n, \dot{\pi}_1, \dots, \dot{\pi}_{n-m}), \quad i = 1, \dots, n$$

with free quasi-velocities $\dot{\pi}_1, \dots, \dot{\pi}_{n-m}$. The quasi-velocities $\dot{\pi}_k(t)$ have corresponding quasi-coordinates $\pi_k(t)$.

We will adopt the standard rule of partial differentiation with respect to the quasi-coordinate π_k

$$\frac{\partial q_i}{\partial \pi_k} = \frac{\partial \dot{q}_i}{\partial \dot{\pi}_k}, \quad \frac{\partial f}{\partial \pi_k} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial \pi_k} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial \dot{q}_i}{\partial \dot{\pi}_k} \tag{1.3}$$

where $f(q_1, \dots, q_n, t)$ is an arbitrary function.

We will reduce the differential constraints to the form

$$\dot{q}_{p+\nu} = \varphi_{p+\nu}(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_p), \quad p = n - m, \quad \nu = 1, \dots, m \tag{1.4}$$

assuming that $\dot{\pi}_i = \dot{q}_i$ $(i = 1, \dots, p)$, and construct the functions

$$T^* = T^*(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_p, t), \quad \varphi^*_{p+\nu}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_p, t)$$

The function T^* is obtained from the kinetic energy of the system T by replacing the velocities $\dot{q}_{p+\nu}$ ($\nu = 1, \ldots, m$) with their expressions in terms of differential constraints, while the function $\varphi^*_{p+\nu}$ are identical in form with the corresponding functions $\varphi_{p+\nu}$. However, the derivatives of the functions T^* and $\varphi^*_{p+\nu}$ are taken taking into account rule (1.3) of differentiation with respect to the quasi-coordinate

$$\frac{\partial q_{p+\nu}}{\partial q_k} = \frac{\partial \dot{q}_{p+\nu}}{\partial \dot{q}_k} = \frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_k}, \quad \nu = 1,\ldots,m, \quad k = 1,\ldots,p$$

The partial derivatives of the functions T and $\varphi_{p+\nu}$ will, as before, be calculated as if all their arguments were independent. Then, the coordinates q_i, defining the motion of the mechanical system, restricted by differential constraints, satisfy [1] the system of equations

$$\frac{d}{dt}\left(\frac{\partial T^*}{\partial \dot{q}_i}\right) - \frac{\partial T^*}{\partial q_i} = Q_i^* + \tilde{Q}_i, \quad i = 1,\ldots,p, \quad p = n - m \tag{1.5}$$

where

$$Q_i^* = Q_i + \sum_{\nu=1}^{m} Q_{p+\nu} \frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_i}, \quad \tilde{Q}_i = \sum_{\nu=1}^{m} \frac{\partial T}{\partial \dot{q}_{p+\nu}} \left[\frac{d}{dt}\left(\frac{\partial \varphi_{p+\nu}}{\partial \dot{q}_i}\right) - \frac{\partial \varphi^*_{p+\nu}}{\partial q_i}\right]$$

The system of equations (1.5) holds whatever constraints are imposed on the system. However, it is not complete. In order to close this system, it is necessary to add kinematic equations (1.4) to it.

2. THE VECTOR EQUATION OF MOTION A NEWTONIAN CENTRAL FIELD

We will take a stationary, right-oriented, orthonormalized frame of reference $O\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ with origin O at the attracting centre. The radius vector of a particle and its velocity vector will be denoted respectively by

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3, \quad \mathbf{v} = \dot{r}_1\mathbf{e}_1 + \dot{r}_2\mathbf{e}_2 + \dot{r}_3\mathbf{e}_3$$

Then, the condition for the velocity modulus to be constant can be represented in the form

$$\mathbf{v}^2 = \dot{r}_1^2 + \dot{r}_2^2 + \dot{r}_3^2 = v_0^2 = \text{const} \tag{2.1}$$

The gravitational force $\mathbf{F}$ and its force function U have the forms

$$\mathbf{F} = -\frac{\mu m}{r^3}\mathbf{r}, \quad U = \frac{\mu m}{r}; \quad r = \sqrt{r_1^2 + r_2^2 + r_3^2}$$

respectively, μ is the gravitational and m is the mass of the particle.

Assuming constraint (2.1) to be ideal, we will examine the equation of motion with the Lagrange multiplier λ. From Eq. (1.2), we obtain

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \lambda\mathbf{v} \tag{2.2}$$

Consequently, the reaction of the ideal constraint is directed along the velocity vector.

The multiplier λ is easy to determine. Differentiating constraint equation (2.1), by virtue of the equation of motion, we obtain

$$0 = \mathbf{v}\dot{\mathbf{v}} = -\frac{\mu}{r^3}\mathbf{r}\cdot\mathbf{v} + \lambda v_0^2, \quad \lambda = \frac{\mu}{r^3 v_0^2}\mathbf{r}\cdot\mathbf{v}$$

Hence, Eq. (2.2) takes the form

$$\dot{\mathbf{v}} = -\frac{\mu}{r^3 v_0^2}[\mathbf{v}\times(\mathbf{r}\times\mathbf{v})]$$

Multiplying both sides of this equation on the left by the vector $\mathbf{r}$, we set up an equation for the change in the specific kinetic moment $\boldsymbol{\sigma} = \mathbf{r} \times \mathbf{v}$ of a particle with respect to the attracting centre O

$$\frac{d\boldsymbol{\sigma}}{dt} = \lambda\boldsymbol{\sigma} = \frac{\mu\boldsymbol{\sigma}}{r^3 v_0^2}(\mathbf{r}\cdot\mathbf{v}) \tag{2.3}$$

It follows that the vector of the kinetic moment retains its orientation in absolute space, while the motion of the particles takes place in a stationary plane perpendicular to this vector. We multiply both sides of Eq. (2.3) scalarly by $\boldsymbol{\sigma}$

$$\frac{1}{2}\frac{d\sigma^2}{dt} = \frac{\mu\sigma^2}{r^2 v_0^2}\left(\frac{\mathbf{r}}{r}\mathbf{v}\right) = \sigma^2\frac{\mu\dot{r}}{r^2 v_0^2}, \quad \dot{r} = \frac{\mathbf{r}\cdot\mathbf{v}}{r}$$

We see that, if at any instant of time $\sigma = 0$ (the velocity is directed along the radius vector), this equality will also hold at any instant of time. If $\sigma \neq 0$, the first integral

$$\ln\frac{\sigma}{\sigma_0} = \frac{1}{v_0^2}\left(\frac{\mu}{r_0} - \frac{\mu}{r}\right), \quad \sigma_0 = \sigma(t_0), \quad r_0 = r(t_0) \tag{2.4}$$

holds, expressing the dependence of the magnitude of the kinetic moment on the distance to the attracting centre.

3. THE EQUATION OF MOTION IN POLAR COORDINATES

We will direct the basis vector $\mathbf{e}_3$ along vector $\boldsymbol{\sigma}$. Then, motion will occur in the plane $O\mathbf{e}_1\mathbf{e}_2$, and the radius vector $\mathbf{r}$ of the particle will take the form

$$\mathbf{r} = r_1\mathbf{e}_1 + r_2\mathbf{e}_2$$

The polar coordinates (r, ϑ) will be chosen such that

$$r_1 = r\cos\vartheta, \quad r_2 = r\sin\vartheta$$

We will use Eqs (1.5), assuming $q_1 = r$ and $q_2 = \vartheta$. The kinetic energy, force function and constraint equation have the form

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\vartheta}^2), \quad U = \frac{\mu m}{r}, \quad \dot{\vartheta} = \frac{\sqrt{v_0^2 - \dot{r}^2}}{r} = \varphi(r, \dot{r})$$

(the plus sign in front of the root is chosen because the direction of the vector $\mathbf{e}_3$ coincides with the direction of the vector $\boldsymbol{\sigma}$). After eliminating the quantity $\dot{\vartheta}$ from the expression for the kinetic energy, we obtain $T^* \equiv m v_0^2/2$. Furthermore

$$\tilde{Q}_r = mr^2\dot{\vartheta}\left[\frac{d}{dt}\left(\frac{\partial\varphi}{\partial\dot{r}}\right) - \frac{\partial\varphi}{\partial r}\right], \quad Q_r^* = -\frac{\mu m}{r^2}$$

We will construct the dynamic equation (1.5)

$$\tilde{Q}_r + Q_r^* = 0 \to r^2\dot{\vartheta}\left[\frac{d}{dt}\left(\frac{\partial\varphi}{\partial\dot{r}}\right) - \frac{\partial\varphi}{\partial r}\right] - \frac{\mu}{r^2} = 0 \tag{3.1}$$

Multiplying this equation by $\dot{r}$ and transforming the expression in square brackets, we obtain

$$r^2\dot{\vartheta}\frac{d}{dt}\left(\dot{r}\frac{\partial\varphi}{\partial\dot{r}} - \varphi\right) - \frac{\mu\dot{r}}{r^2} = 0$$

We will take into account that $\sigma = r^2\dot{\vartheta} = r\sqrt{v_0^2 - \dot{r}^2}$. Then, the latter equation reduces to the form

$$\frac{1}{\sigma}\frac{d\sigma}{dt}=\frac{\mu\dot{r}}{r^2 v_0^2}$$

and yields the first integral (2.4) already obtained above.

4. REDUCTION OF THE SOLUTION TO QUADRATURES

We will first consider steady motions. The first of these is the motion of a particle along a ray emanating from the attracting centre. For such motion $\sigma \equiv 0$. As already pointed out, this solution exists. To obtain it, it is sufficient for particle velocity to be collinear with the initial radius vector, and the particle will move at a constant velocity along the corresponding ray in a direction either away from the attracting centre or towards the attracting centre.

The second steady motion is the motion of a particle in a circle with constant velocity v_0. In order to obtain such a solution, we will consider Eq. (3.1). In it

$$\frac{\partial\varphi}{\partial\dot{r}}=-\frac{\dot{r}}{r\sqrt{v_0^2-\dot{r}^2}},\quad \frac{\partial\varphi}{\partial r}=-\frac{\sqrt{v_0^2-\dot{r}^2}}{r^2}$$

Assuming now that in Eq. (3.1) $\dot{r}\equiv 0$ (the direction of the velocity is perpendicular to the radius vector), we find the condition for the initial value of the radius to agree

$$v_0^2=\frac{\mu}{r_0}$$

In other words, the initial value of the radius should be such that the velocity v_0 is the first space velocity for it. Note that with such motion in a circle $\lambda \equiv 0$, and there is no reaction of the constraint.

To investigate unsteady motions, we will use the first integral (2.4), which we will represent in the form

$$\sqrt{1-y^2}=\frac{f(x)}{a},\quad f(x)=xe^{-x},\quad y=\frac{\dot{r}}{v_0},\quad x=\frac{\mu}{v_0^2 r},\quad a=\frac{f(x_0)}{\sqrt{1-y_0^2}} \tag{4.1}$$

The region in which the variable x defined is expressed by the inequalities

$$f(x)=xe^{-x}\leqslant a,\quad x>0 \tag{4.2}$$

When $x > 0$, the function $f(x)$ is positive and has a unique maximum $f(1) = e^{-1}$. Furthermore, $f(x)\to 0$ when $x\to 0$ or $x\to\infty$. When $x = 2$, it has a point of inflection.

When $\dot{r}\neq 0$, the following formulae hold

$$|\dot{r}|=v_0 g(x),\quad \dot{x}=-\frac{\mu(\operatorname{sign}\dot{r})}{v_0 r^2}g(x),\quad \frac{d\vartheta}{dx}=-\frac{v_0}{a\dot{r}}e^{-x};\quad g(x)=\sqrt{1-\frac{f^2(x)}{a^2}} \tag{4.3}$$

Therefore, in the region in which the sign of $\dot{r}$ is constant, we have the quadratures

$$t=t_0-\frac{\mu a(\operatorname{sign}\dot{r})}{v_0^3}\int_{x_0}^{x}\frac{dx}{x^2\sqrt{a^2-x^2e^{-2x}}},\quad \vartheta=\vartheta_0-(\operatorname{sign}\dot{r})\int_{x_0}^{x}\frac{e^{-x}dx}{\sqrt{a^2-x^2e^{-2x}}} \tag{4.4}$$

5. QUALITATIVE INVESTIGATION OF THE MOTION

Solving Eq. (4.1) for y, we obtain in the (x, y) plane a family of curves with respect to the parameter a that form the phase diagram (Fig. 1) of the radial component of the motion

$$|y|=g(x) \tag{5.1}$$

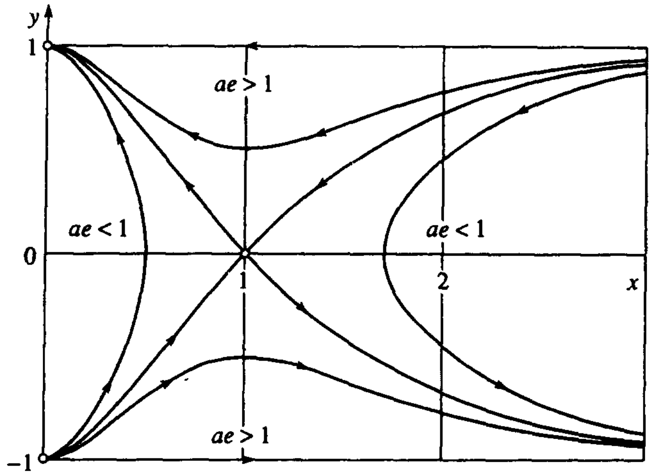

Fig. 1

Depending on the magnitude of a, the following cases are possible.

5.1. *The case* $ae > 1$: $ef(x_0) > \sqrt{1 - y_0^2}$. This means that the modulus of the radial component of the initial velocity is fairly high. The variable y does not vanish whatever the value of the radius. Due to the fact that $f(x) > 0$, the transverse component of the velocity does not vanish either. If $\dot{r}_0 < 0$, the particle moves towards the attracting centre and $x \to \infty$, but, if $\dot{r}_0 > 0$, it moves away from the attracting centre and $x \to 0$. In this case, the polar radius changes monotonically, and the angular velocity of the radius vector tends to zero. The radius vector of the particle may make several turns about the attracting centre. However, the number of turns will not be infinite.

We will show this. Note that, since $f(x) \leqslant e^{-1}$, we have

$$|y| \geqslant g(1), \quad |\dot{r}| \geqslant v_0 g(1); \quad g(1) = \sqrt{1 - (ae)^{-2}}$$

We will consider some versions of this.

5.1.1. The version $\dot{r}_0 > 0$: the radius increases monotonically, and x decreases monotonically. Therefore

$$\vartheta - \vartheta_0 = \int_x^{x_0} \frac{v_0}{a\dot{r}} e^{-x} dx \leqslant \int_x^{x_0} \frac{e^{-x}}{ag(1)} dx < \frac{1 - e^{-x_0}}{ag(1)}$$

5.1.2. The version $\dot{r}_0 < 0$: the radius decreases monotonically, and x increases monotonically. Therefore

$$\vartheta - \vartheta_0 = \int_{x_0}^{x} \frac{v_0}{a|\dot{r}|} e^{-x} dx \leqslant \int_{x_0}^{x} \frac{e^{-x}}{ag(1)} dx < \frac{e^{-x_0}}{ag(1)}$$

5.2. The case $ae = 1$: $ef(x_0) = \sqrt{1 - y_0^2}$. For such initial conditions, y vanishes when $x = 1$, to which value the radius of the circular orbit of motion at velocity v_0 corresponds. In Fig. 1, the separatrix passing through the point (1, 0) corresponds to this case. The separatrix intersects the Ox axis at an angle $\pi/4$. In order to demonstrate this, we will put $z = x - 1$. Then, from Eq. (5.1), for $a = e^{-1}$ and small $|z|$, we have

$$|y| = \sqrt{1 - (1+z)^2 e^{-2z}} \approx |z| \sqrt{1 - 2z - 2z^2 + o(z^2)}$$

We will investigate the motion along the separatrix. The following versions can be distinguished.

5.2.1. The version $0 < x_0 < 1$, $\dot{r}_0 > 0$. The initial point and the attracting centre are on different sides of the circular orbit. The radial component of the velocity $\dot{r}$ and the radius r increase monotonically,

$x \to 0$, and the particle departs to infinity, performing, perhaps, a finite number of turns about the attracting centre. In fact

$$\vartheta - \vartheta_0 = \int_x^{x_0} \frac{v_0}{a\dot{r}} e^{-x} dx \leqslant \int_x^{x_0} \frac{v_0}{a\dot{r}_0} e^{-x} dx > \frac{v_0}{a\dot{r}_0}(1 - e^{-x_0}) \tag{5.2}$$

5.2.2. The version $0 < x_0 < 1, \dot{r}_0 < 0$. Then $\dot{r} \to 0$ monotonically, $x \to (1-0)$, and the radius decreases, and here $r \to \mu/v_0^2$. Thus, the trajectory approaches a circular orbit about the attracting centre. We will show that $t \to \infty$ when $x \to (1-0)$. For this version the following formula holds [5]

$$\dot{x} = \frac{v_0^3 x^2 g(x)}{\mu} \leqslant \frac{1-x}{h}, \quad h = \frac{\mu}{v_0^3 \sqrt{1 - e(x_0 - 1)}}$$

Therefore

$$t - t_0 \geqslant h \int_{x_0}^{x} \frac{dx}{1-x} = h \ln \frac{1-x_0}{1-x}$$

Consequently, when $x \to (1-0)$, the time of motion along the trajectory increases without limit, and the trajectory coils from the outside on to a circular orbit with an infinite number of turns about the attracting centre.

5.2.3. The version $1 < x_0 < +\infty, \dot{r}_0 < 0$. The initial point and the attracting centre are on the same side of the circular orbit. The radial component of the velocity is negative. Therefore, the polar radius of the particle will decrease monotonically, the radial component of velocity, remaining negative, will increase in absolute magnitude, and the particle will approach the attracting centre without limit, and here the direction of the velocity will asymptotically tend towards the direction opposite to the direction of the radius vector. The radius vector of the particle can make only a finite number of turns about the attracting centre:

$$\vartheta - \vartheta_0 = \int_{x_0}^{x} \frac{v_0}{a|\dot{r}|} e^{-x} dx \leqslant \int_{x_0}^{x} \frac{v_0}{a|\dot{r}_0|} e^{-x} dx < \frac{v_0}{a|\dot{r}_0|} e^{-x_0} \tag{5.3}$$

5.2.4. The version $1 < x_0 < +\infty, \dot{r}_0 > 0$. The initial point and attracting centre are on the same side of the circular orbit. The radial component of the velocity is positive. Therefore, the polar radius of the point will increase monotonically, approaching the radius of the circular orbit. Here, $x \to 1 + 0$, and the radial component of the velocity will approach zero. For the case under examination

$$\dot{r} = v_0 g(x), \quad \dot{x} = -\frac{v_0^3 x^2 g(x)}{\mu}$$

Here, at lease for $x \leqslant 3$, the following inequality holds [5]

$$f(x) \geqslant e^{-1}[1 - (x-1)^2/2]$$

Therefore

$$t - t_0 \geqslant \frac{\mu}{v_0^3 x_0^2} \int_x^{x_0} \frac{dx}{x-1} = \frac{\mu}{v_0^3 x_0^2} \ln \frac{x_0 - 1}{x - 1}$$

Thus, the time taken to reach the value $x = 1$ proves to be infinite, and the particle moves in an orbit having an infinite number of turns and coiling into circular orbit from the side of the attracting centre.

5.3. *The case $ae < 1$*: $ef(x_0) < \sqrt{1 - y_0^2}$. This means that the modulus of the initial radial component of the velocity is comparatively small. Now, the equation

$$f(x) = xe^{-x} = a \tag{5.4}$$

has two real roots: $x_\pi < 1$, $x_\alpha > 1$: $f(x) = f(x_a) = a$, so that the region in which (4.2) is defined is the combination of two half-intervals $(0, x_\pi] \cup [x_a, +\infty)$. We will consider corresponding versions.

5.3.1. The version $0 < x_0 < x_\pi < 1, \dot{r}_0 > 0$. The initial point and the attracting centre are on different sides of the circular orbit. During motion, the particle moves away to infinity, the radial component of the velocity increases, and the number of turns of its trajectory in this case in finite, since limit (5.2) holds.

5.3.2. The version $0 < x_0 < x_\pi < 1, \dot{r}_0 < 0$. The initial point and the attracting centre lie on different sides of the circular orbit. During motion, the polar radius decreases, $x \to x_\pi$, and the radial velocity, being negative, increases monotonically to zero. The time taken to reach the value $x = x_\pi$ is finite, since

$$f(x_\pi) = a, \quad f'(x) = (1-x)e^{-x} > 0, \quad f''(x) = -(2-x)e^{-x} < 0$$

and the following inequality holds [5]

$$t - t_0 \leqslant \frac{2\mu}{v_0^3 x_0^2}\sqrt{\frac{a(x_\pi - x_0)}{f'(x_\pi)}}$$

Furthermore, from formulae (4.3) we find

$$\ddot{r} = v_0 \frac{f(x)f'(x)\dot{x}}{a^2 g(x)} = \frac{v_0^4}{\mu a^2} x^2 f(x) f'(x) \tag{5.5}$$

In the present case $f'(x_\pi) > 0$. Therefore, $\ddot{r}(x_\pi) > 0$, and the velocity of the point, reaching a zero value when $x = x_\pi$, will become positive, while the polar radius will begin to increase without limit.

When $x \to (x_\pi - 0)$, the particle may make only a finite number of turns around the attracting centre. In fact

$$\vartheta - \vartheta_0 \leqslant \int_x^{x_\pi} \frac{v_0 dx}{ag(x)} \leqslant \frac{2v_0}{\sqrt{af'(x_\pi)}}\sqrt{x_\pi - x}$$

5.3.3. The version $x_a < x_0 < +\infty$, $\dot{r}_0 < 0$. At the initial instant of time, the particle and attracting centre lie on the same side of the circular orbit. During motion, $x \to \infty$ and the polar radius decreases monotonically, but the modulus of the radial component of the velocity increases monotonically, the direction of velocity tends asymptotically towards the direction opposite to the direction of radius vector, and the particle moves towards the attracting centre, making, perhaps, a finite number of turns around it, since estimate (5.3) holds.

5.3.4. The version $x_\alpha < x_0 < +\infty, x_\alpha < 2, \dot{r}_0 > 0$. At the initial instant of time, the particle and the attracting centre lie on the same side of the circular orbit. The polar radius initially increases, but $x \to x_\alpha$, where the radial velocity vanishes. Since, when $x < 2$, the function $f(x)$ is convex upwards, inequalities similar to the limits of version 5.3.2 hold. Consequently, the variable x reaches a value x_α after a finite time. According to formula (5.5), $\ddot{r}(x_\alpha) < 0$. Therefore, the radial velocity of the particle, passing through zero, will become negative, the polar radius will begin to decrease, and the particle will approach without limit the attracting centre after no more than a finite number of turns around it in accordance with limit (5.3).

5.3.5. The version $x_\alpha < x_0 < +\infty, x_\alpha \geqslant 2, \dot{r}_0 > 0$. At the initial instant of time, the particle the attracting centre lie on the same side of the circular orbit. The radius initially increases, and $x \to (x_\alpha + 0)$, where the radial component of the velocity vanishes. However, now the function $f(x)$ is convex downwards. Consider the neighbourhood $x_\alpha < x_0 < x_\alpha + 1$. For it [5]

$$t - t_0 \leqslant \frac{2\mu}{v_0^3 x_\alpha^2}\sqrt{\frac{a}{f(x_\alpha) - f(x_\alpha + 1)}}(\sqrt{x_0 - x_\alpha} - \sqrt{x - x_\alpha})$$

and the time taken to reach the value $x = x_\alpha$ is finite. When $x = x_\alpha$, we have $\ddot{r}(x_\alpha) < 0$. As in the previous case, the radial velocity, passing through zero, becomes negative, and the radius, tends to zero without limit. The particle may perform no more than a finite number of turns around the attracting centre in accordance with estimate (5.3).

The phase diagram in Fig. 1 contains two more straight lines: $y = 1$ and $y = -1$. Both correspond to motion along the radius vector at a constant velocity v_0. On the line $y - 1$, the particle departs to infinity, and the line $y = -1$ the particle approaches the attracting centre without limit.

6. SOME PROBLEMS OF MANOEUVRING

We will consider problems associated with the need to ensure the required boundary conditions.

6.1. *Injection into a near-circular orbit.* Suppose the radius of the circular orbit, r_c, is given, and the initial polar radius of the point $r_0 \neq r_c$. As follows from an analysis of the results of Section 5.2, there is no phase trajectory leading from the position with radius r_0 to a circular orbit in a finite time. However, it is possible to indicate a trajectory which approaches infinitesimally closely to the prescribed circular orbit. We will supplement the initial conditions such that the initial point of motion on the phase diagram (Fig. 1) belongs to a branch of the separatrix leading to the point (1, 0). To do this we will put

$$v_0 = \sqrt{\frac{\mu}{r_c}}, \quad x_0 = \frac{\mu}{v_0^2 r_0}, \quad \dot{r}_0 = V_0 \operatorname{sign}(x_0 - 1), \quad V_0 = v_0\sqrt{1 - e^2 f^2(x_0)}$$

Then, obviously, $a = f(x_0)/\sqrt{1 - y_0^2} = e^{-1}$, and we obtain the solution of the problem.

6.2. *Attainment of the prescribed value of the polar radius.* Suppose two values of the polar radius, r_0 and r_1, are given, and also the velocity v_0. It is required to find the entire set of trajectories along which, starting from the point with polar radius r_0, it is possible to reach the point with polar radius r_1.

The following values correspond to the polar radii

$$x_0 = \frac{\mu}{v_0^2 r_0}, \quad x_1 = \frac{\mu}{v_0^2 r_1}$$

and we will use the phase diagram in Fig. 1. Depending on the values of x_0 and x_1, we will consider the following versions.

6.2.1. The version $0 < x_1 < x_0 < 1$. It is obvious that, then, $r_1 > r_0 > r_c$, and for any value of $\dot{r}_0$, from the half-interval $(-V_0, v_0]$, the polar radius r_1 is reached, and other values of $\dot{r}_0$ cannot serve as a solution. Here, the lower limit of the region of solutions $\dot{r}_0$ corresponds to the separatrix leading to the point (1.0) in Fig. 1.

6.2.2. The version $0 < x_1 \leqslant 1 \leqslant x_0$. We have $r_1 \geqslant r_c \geqslant r_0$, and it is impossible to attain the value of the radius r_1 from the region of negative values of the variable y in Fig. 1. The solution of the problem is written in the form of the inequality

$$V_0 < \dot{r}_0 \leqslant v_0$$

The lower limit represents the point on the separatrix incident on the point (1, 0) from above.

6.2.3. The version $1 < x_1 < x_0$. In other words, $r_c > r_1 > r_0$. The value of the radius r_1 can be attained only from the region of positive values of the variable y. The solution of the problem is given by the inequality

$$V_1 \leqslant \dot{r}_0 \leqslant v_0$$

in which the lower limit

$$V_1 = v_0\sqrt{1 - \frac{f^2(x_0)}{f^2(x_1)}}$$

corresponds to the phase curve incident from above at the point $(x_1, 0)$ in Fig. 1. If the inequality

$$V_1 < \dot r_0 < V_0$$

is satisfied, the value r_1 will be reached twice: the first time on the ascending section of the trajectory, and the second time on the descending section.

6.2.4. The version $0 < x_0 < x_1 < 1$. We have $r_0 > r_1 > r_c$. The polar radius r_1 can be reached only when

$$-v_0 \leqslant \dot r_0 \leqslant -V_1$$

The upper limit in this inequality corresponds to the phase curve intersecting the Ox axis "from below" at the point $(x_1, 0)$. If the inequality

$$-V_0 < \dot r_0 < -V_1$$

is satisfied, the value of the polar radius r_1 will be attained twice: first on the descending section and then on the ascending section of the trajectory.

6.2.5. *The version* $0 < x_0 \leqslant 1 \leqslant x_1$. We have $r_0 \geqslant r_c \geqslant r_1$. The polar radius r_1 is attained only if the condition

$$-v_0 \leqslant \dot r_0 < -V_0$$

is satisfied. The upper limit corresponds to the point on the separatrix incident at the point $(1, 0)$ "from below".

6.2.6. The version $1 < x_0 < x_1$. Then $r_c > r_0 > r_1$. The radius r_1 can be reached only if the inequality

$$-v_0 \leqslant \dot r_0 < V_0$$

is satisfied. The upper limit corresponds to the separatrix incident at point $(1, 0)$ in Fig. 1 "from above".
The versions examined exhaust the solution of problem 6.2.

6.3. *Attainment of the radius with the required radial velocity*. The values of the polar radius r_0 and r_1 and also the velocity v_0 are specified. It is required to find the trajectory for which, starting from the point with polar radius r_0, the polar radius r_1 is reached with the prescribed radial velocity $\dot r_1$.
The solution of this problem can be obtained by the time inversion. Suppose the problem is solved, and t_1 is the instant at which the radius r_1 is reached. We will introduce the independent variables $\tau = t_1 - t$. In inverse time τ, the radius r_1 corresponds to the instant $\tau = 0$ with a prescribed initial radial component of the velocity $r_0' = -\dot r$. The polar radius r_0 becomes finite. It is now possible to use the results of the solution of problem 6.2, making the substitution $x_0 = \mu/(v_0^2 r_1)$ and $x_1 = \mu/(v_0^2 r_0)$. A solution of the problem exists if, for the corresponding case of the position of the numbers x_0 and x_1 on the Ox axis in Fig. 1, the prescribed quantity r_0' falls in the region of the solution of problem 6.2. Then, r_0' in Fig. 1 specifies the unique curve $\{y(\tau), x(\tau)\}$. For it we obtain

$$y_0 = \frac{r_0'}{v_0} = -\frac{\dot r_1}{v_0}, \quad a = \frac{f(x_0)}{\sqrt{1 - y_0^2}}$$

The points of the curve for which $x = x_1$ will yield the solution of the problem, which is given by the formulae

$$|y_1| = g(x_1), \quad r_1' = \pm v_0 |y_1|$$

The sign of the solution is chosen depending on the branch of the curve of the phase diagram that gives the solution. After this, finally we have $\dot r_0 = -r_1'$.
Note that the quality t_1 is secondary in nature and is introduced only for convenience. Its actual value plays no part in the solution procedure. Note also that problem 6.3 may have no more than two solutions.

6.4. *Calculation of the angular distance*. Let us assume that the initial value and the final value of the polar radius, r_0 and r_1, the magnitude of the velocity v_0, and the initial radial component of the velocity,

$\dot{r}_0$, are specified, and that $|\dot{r}_0| \leqslant v_0$. We will denote by ϑ_0 and ϑ_1 respectively the initial and final values of the angular coordinate ϑ. It is required to find the angular distance $\theta = \vartheta_1 - \vartheta_0$ during motion at the specified velocity v from the initial point with radius r_0 to the final point with radius r_1.

We note to begin with that, if $|\dot{r}_0| = v_0$, motion occurs along a ray emanating from the attracting centre, and then $\theta = 0$. Subsequently, we will assume that $|\dot{r}_0| < v_0$. We will adopt the quantities

$$x_0 = \frac{\mu}{v_0^2 r_0}, \quad x_1 = \frac{\mu}{v_0^2 r_1}, \quad y_0 = \frac{\dot{r}_0}{v_0}, \quad a = \frac{f(x_0)}{\sqrt{1-y_0^2}}$$

and introduce the function

$$\Theta(\xi_1, \xi_2, a) = \int_{\xi_1}^{\xi_2} \frac{e^{-x}dx}{\sqrt{a^2 - x^2 e^{-2x}}}$$

We will examine the following versions.

6.4.1. The version $ae < 1, x_0 \leqslant x_\pi, x_1 < x_\pi$. Motion can occur if the conditions

$$(x_1 \leqslant x_0 \leqslant x_\pi) \& (y_1 \geqslant 0) \quad (x_0 \leqslant x_1 \leqslant x_\pi) \& (y_0 \leqslant 0)$$

are satisfied, and the angular distance is expressed by the formula [5]

$$\theta = \Theta_\pi(x_1, a)\operatorname{sign} y_1 - \Theta_\pi(x_0, a)\operatorname{sign} y_0, \quad \Theta_\pi(x, a) = \Theta(x, x_\pi, a) \tag{6.1}$$

6.4.2. The version $ae < 1, x_0 \geqslant x_\alpha, x_1 > x_\alpha$. Motion occurs if the conditions

$$(x_\alpha \leqslant x_0 \leqslant x_1) \& (y_1 \leqslant 0) \quad (x_\alpha \leqslant x_1 \leqslant x_0) \& (y_0 \geqslant 0)$$

are satisfied, and the angular distance is expressed by the formula [5]

$$\theta = \Theta_\alpha(x_0, a)\operatorname{sign} y_0 - \Theta_\alpha(x_1, a)\operatorname{sign} y_1, \quad \Theta_\alpha(x, a) = \Theta(x_\alpha, x, a) \tag{6.2}$$

6.4.3. The version $ae = 1, x_0 < 1, x_1 < 1$. The representative point in Fig. 1 can only move along the separatrix. The region in which the solution can exist is expressed by the inequalities

$$(x_1 \leqslant x_0 < 1) \& (y_0 > 0) \quad (x_0 \leqslant x_1 < 1) \& (y_0 < 0)$$

and the angular distance is calculated from the formula

$$\theta = [\Theta_0(x_0, a) - \Theta_0(x_1, a)]\operatorname{sign} y_0, \quad a = e^{-1}, \quad \Theta_0(x, a) = \Theta(0, x, a) \tag{6.3}$$

(if $ae = 1$, then $\Theta(x, a) \to \infty$ when $x \to 1$).

6.4.4. The version $ae = 1, x_0 > 1, x_1 > 1$. As in the case of Section 6.4.3, the representative point only moves along the separatrix. The region in which the solution exists is expressed by the inequalities

$$(1 < x_1 \leqslant x_0) \& (y_0 > 0) \quad (1 < x_0 \leqslant x_1) \& (y_0 < 0)$$

and the angular distance is calculated from the formula

$$\theta = [\Theta_\infty(x_1, a) - \Theta_\infty(x_0, a)]\operatorname{sign} y_0, \quad a = e^{-1}, \quad \Theta_\infty(x, a) = \Theta(x, \infty, a) \tag{6.4}$$

Despite the fact that the function $\Theta_\infty(x, a)$ is an improper integral, for any $x \neq 1$ it takes a finite value; here, $\Theta\infty(x, a) \to \infty$ when $x \to 1$.

6.4.5. The version $ae > 1$. Motion of the representative point along the phase curve occurs without any singularities, and it possible to use any of the functions $\Theta_0(x, a)$ and $\Theta_\infty(x, a)$, which are connected by the obvious relation

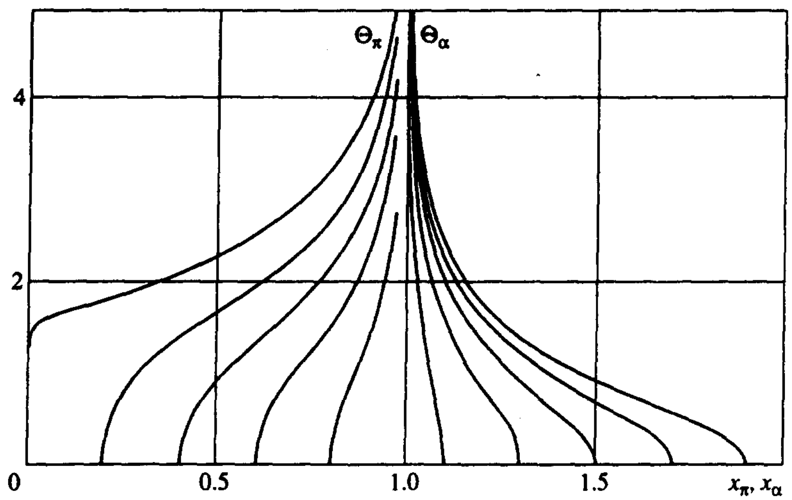

Fig. 2

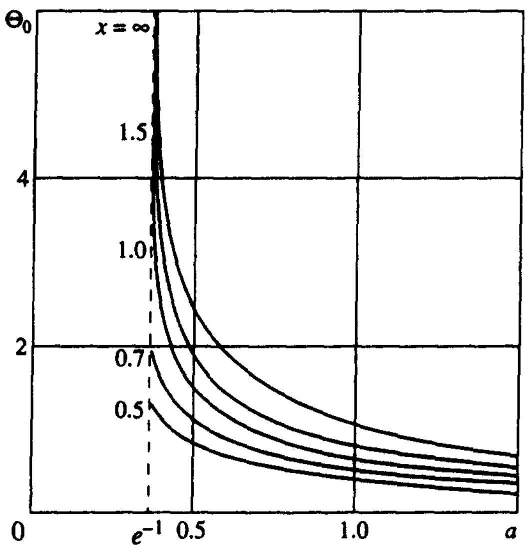

Fig. 3

$$\Theta_0(x,a)+\Theta_\infty(x,a)=\Theta(\infty,a)=\Theta_\infty(0,a)=\Theta(0,\infty,a) \tag{6.5}$$

The region in which the solution exists is defined by the inequalities

$$(x_1 \leqslant x_0) \,\&\, (y_0 > 0), \quad (x_0 \leqslant x_1) \,\&\, (y_0 < 0)$$

and the angular distance is calculated from the formula

$$\theta = [\Theta_0(x_0,a)-\Theta_0(x_1,a)]\operatorname{sign} y_0 = [\Theta_\infty(x_1,a)-\Theta_\infty(x_0,a)]\operatorname{sign} y_0$$

This case exhausts the solution of problem 6.4.

7. PROPERTIES OF THE FUNCTIONS EXPRESSING THE ANGULAR DISTANCE

We will first consider the function Θ_π, defined by the final formula of (6.1). In it, x_π is the least root of Eq. (5.4), so that $a = x_\pi e^{-x_\pi}$. We will investigate the behaviour of the function Θ_π in a small neighbourhood of the point x. Suppose $x = x_\pi - y$ and $y \geqslant 0$. Then, we obtain

$$\Theta_\pi = \frac{1}{x_\pi}\int_0^y \frac{e^y dy}{\sqrt{1-(1-y/x_\pi)^2 e^{2y}}}$$

We will assume that $y \leqslant 1$, and retain in the numerator and the radicand of the denominator only terms that are linear in y. After integration and changing to the initial variable x, we obtain

$$\Theta_\pi \approx \frac{\sqrt{2}}{\sqrt{x_\pi(1-x_\pi)}}\left[\sqrt{x_\pi - x} + \frac{1}{3}(x_\pi - x)^{3/2}\right] \tag{7.1}$$

From this it is clear that, for a fixed value of x, the function Θ_π vanishes at the point $x_\pi = x$ and has a vertical tangent.

The limit of function (7.1) when x $\to$ 0 has the form $\Theta_\pi^* = \sqrt{2}\,(1 + x_\pi/3)\sqrt{1 - x_\pi}$. At the point $x_\pi = 0$ we obtain $\Theta_\pi^* = \sqrt{2}$ and $d\Theta_\pi^*/dx_\pi = \sqrt{2}/3$.

When $x_\pi \to 1$ the phase curves in Fig. 1 approach separatrices. Motion along the separatrix leading to the point (1, 0) is accompanied by an infinite increase in the angular distance. Therefore, but fixed x, the function Θ_π has a vertical asymptote $x_\pi = 1$ and tends to $+\infty$ when $x_\pi \to 1 - 0$.

Figure 2, in the region $0 < x_\pi < 1$, shows graphs of the function Θ_π obtained numerically. The parameter here is x, and the argument is x_π. The value of x for the separate graph corresponds to the point of intersection of the graph with the abscissa axis.

The behaviour of the function Θ_α, defined by the final formula of (6.2), is similar in many ways to the behaviour of the function Θ_π. It can be investigated analytically by a similar methods. Figure 2, in the region $x_\alpha > 1$, shows a family of functions Θ_π found numerically. The parameter of the family is x. The value of x corresponds to the intersection of the graph with the abscissa axis.

The functions Θ_α and Θ_π examined above are convenient for calculating the distance in the case when $a < e^{-1}$. When $a = e^{-1}$, the function $\Theta_0(x, a)$, defined by the final formula of (6.3), should be used in the region $x < 1$, and the function $\Theta_\infty(x, a)$, defined by the final formula of (6.4), should be used in the region $x > 1$. Both functions of the variable x are positive and have a vertical asymptote when $x \to 1$.

When $a > e^{-1}$, any of the functions Θ_0 and Θ_∞ is defined in the entire region of variation of x and is therefore suitable for calculating the angular distance. Figure 3 shows a family of functions Θ_0, where the argument is the variable a and the parameter is x. Physically, this function expresses the angular distance between the infinitely remote point of the trajectory and the point with a radius corresponding to the given value of x. We see that this distance decreases monotonically as a increases. The curve corresponding to $x = \infty$ limits the region of variation of the function Θ_0. The values of the function Θ_∞ can be found from the values of the function Θ_0 using (6.5).

This research was supported financially by the Russian Foundation for Basic Research (01-01-00079, 96-15-96022) and the Federal Special-Purpose "Integration" Programme (1.6–329, A0097).

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Translated by P.S.C.